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A Separation Property of Positive Definite Functions on Locally Compact Groups and Applications to Fourier Algebras[1]

Eberhard Kaniuth

Fachbereich Mathematik/Informatik, Universität Paderborn, D-33095 Paderborn, Germany
E-mail: kaniuth@math.uni-paderborn.de

and

Anthony T. Lau

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For a closed subgroup H of a locally compact group G consider the property that the continuous positive definite functions on G which are identically one on H separate points in $G\backslash H$ from points in H. We prove a structure theorem for almost connected groups having this separation property for every closed subgroup. Also, when a pair (G, H) has this separation property, there are interesting consequences in the ideal theory of the Fourier algebra of G. © 2000 Academic Press

INTRODUCTION

Let G be a locally compact group and $P(G)$ the set of continuous positive definite functions on G. For a closed subgroup H of G, let

$$P_H(G) = \{\phi \in P(G) : \phi(h) = 1 \text{ for all } h \in H\}.$$

We say that G has the *H-separation property* if for every $x \in G$, $x \notin H$, there exists $\phi \in P_H(G)$ such that $\phi(x) \neq 1$. When G has the H-separation property for every closed subgroup H of G, we refer to G as a group with the *separation property*. This property first appeared in [19], where it was noticed that if H is either normal, or compact, or open in G, then G has

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the H-separation property. Moreover, every group with small invariant neighbourhoods has the separation property [11]. This paper is concerned with a more systematic investigation of the separation property and with applications to ideal theory of Fourier algebras.

In Section 1 we study almost connected groups with the separation property, and we identify them as precisely those locally compact groups that contain a normal subgroup of finite index which is a direct product of a vector group and a compact group (Theorem 1.1). Thus, at least for almost connected groups, the requirement to share the separation property is fairly restrictive. Therefore, in Section 2, we provide some more classes and examples of pairs (G, H) such that G has the H-separation property.

Let $A(G)$ be the Fourier algebra of G and, for a closed subgroup H of G, let $I(H)$ denote the closed ideal consisting of all functions in $A(G)$ that vanish on H. In Section 3 we first characterize the H-separation property in terms of the existence of certain projections from the von Neumann algebra $VN(G)$, the dual of $A(G)$, onto the annihilator $I(H)^{\perp}$ (Proposition 3.1). We then apply this characterization to two different problems in the ideal theory of $A(G)$. Suppose that G has the H-separation property. First, when G is amenable, we prove the existence of an approximate identity in $I(H)$ with norm bound 2 (Theorem 3.4). Actually, when G/H is infinite, 2 is the best possible such bound (Theorem 3.4). This improves results from [4, 10, and 11]. Second, we establish, in the general context of $A(G)$-invariant subspaces X of $VN(G)$ [17], an injection theorem for X-Ditkin sets (Theorem 3.5). In particular, this yields injection theorems for Ditkin sets and for local Ditkin sets, respectively.

1. THE SEPARATION PROPERTY FOR ALMOST CONNECTED GROUPS

In this section we completely characterize the almost connected locally compact groups with the separation property by proving the following structure theorem.

THEOREM 1.1. *Let G be an almost connected locally compact group. Then the following two conditions are equivalent.*

(i) *For every closed subgroup H of G and $x \in G, x \notin H$, there exists $\phi \in P_H(G)$ such that $\phi(x) \neq 1$.*

(ii) *G contains an open normal subgroup N of finite index such that N is a direct product of a compact group and a vector group.*

Notice first that if G is as in (ii) (and not necessarily almost connected), then G is an SIN-group, that is, G has a neighbourhood basis $\mathscr{V}$ of the identity such that $x^{-1}Vx = V$ for all $V \in \mathscr{V}$ and $x \in G$. Moreover, every SIN-group does have the separation property by [11, Proposition 3.10]. Thus, Theorem 1.1 in particular shows that an almost connected group has the separation property if and only if it is an SIN-group. Also, it is obvious that if a locally compact group has the separation property, then so does every closed subgroup and every quotient group of G.

To prove (i) $\Rightarrow$ (ii), suppose temporarily that we have already shown that a connected Lie group with separation property is a direct product of a vector group and a compact group. Now, let G be an almost connected group with the separation property. Then G is a projective limit of Lie groups G_α [23, Theorem 4.6], and each G_α has the separation property. Thus the connected component of G_α, which is of finite index in G_α, is a direct product of a compact group and a vector group. In particular, each G_α is an SIN-group. Now, it is well-known and easy to verify that a projective limit of SIN-groups is an SIN-group. Hence, by Theorem 2.13 of [13], G has an open normal subgroup N such that N is the direct product of a compact group and a vector group. Finally, since G is almost connected, N has finite index in G.

It therefore suffices to prove (i) $\Rightarrow$ (ii) for connected Lie groups. To that end, we treat four special cases and then combine these, using structure theory, to establish (i) $\Rightarrow$ (ii) for general connected Lie groups.

To start with, let G be any locally compact group and let H be a closed subgroup of G and $\phi \in P_H(G)$. Then, by [15, (32.6)],

$$\phi(h_1 x h_2) = \phi(x)$$

for all $x \in G$ and $h_1, h_2 \in H$. The basic idea in proving the theorem in special cases is exploiting this property of functions in $P_H(G)$ for appropriate choices of H.

Lemma 1.2. *Let G be a locally compact group containing a closed normal vector subgroup V such that G/V is compact and connected. If G has the separation property, then there exists a compact subgroup K of G such that G is the direct product of K and V.*

Proof. Since G/V is compact and V is a vector group, there exists a compact subgroup K of G such that G is a semi-direct product of K and V [16, Theorem VIII]. Let $\alpha: k \to \alpha_k$ denote the homomorphism from K into $GL(V)$ defining this semi-direct product. Thus $(k, u)(l, v) = (kl, u + \alpha_k(v))$ for all $k, l \in K$ and $u, v \in V$. Let $\langle \cdot, \cdot \rangle$ be any scalar product of V. Replacing $\langle \cdot, \cdot \rangle$ by the new scalar product

$$(u, v) \to \int_K \langle \alpha_k(u), \alpha_k(v) \rangle \, dk,$$

we can henceforth assume that K acts on V by orthogonal transformations.

We proceed by induction on the dimension of V. If $\dim V = 1$, then $\alpha_k = \mathrm{id}_V$ for all $k \in K$ since K is connected. Suppose the statement of the lemma holds whenever the vector group is d-dimensional, and let $\dim V = d+1$. Choose a linear subspace W of V of codimension 1, and let

$$K_W = \{k \in K : \alpha_k(W) \subseteq W\} = \{k \in K : \alpha_k(W) = W\}.$$

Then K_W is a closed subgroup of K, and $\alpha_k(W^\perp) = W^\perp$ for each $k \in K$.

Now, suppose that $K_W \neq K$. For every $k \notin K_W$, we have that $W + \alpha_k(W) = V$ since W is of codimension one. If $\phi \in P_W(G)$, then for all $u, w \in W$,

$$\phi(k, 0) = \phi((e, u)(k, 0)(e, w)) = \phi(k, u + \alpha_k(w)).$$

Thus $\phi(k, 0) = \phi(k, v)$ for all $v \in V$ whenever $k \in K \backslash K_W$. Since $K_W \neq K$ and K is connected, $e \in \overline{K \backslash K_W}$. Continuity of ϕ implies that $1 = \phi(e, 0) = \phi(e, v)$ for all $v \in V$. This contradicts the separation property.

Thus $K_W = K$. Then, by the inductive hypothesis, the semi-direct product of K and W is in fact a direct product $K \times W$. Moreover, since $W^\perp$ is one-dimensional and K is connected, $\alpha_k(v) = v$ for all $v \in W^\perp$. It follows that $G = K \times V$. ∎

We now present three examples of 2-step solvable, simply connected Lie groups of dimension $\leqslant 3$. The failure of the separation property for each of these examples will subsequently be used to show that no non-abelian, solvable, simply connected Lie group can have the separation property.

EXAMPLE 1.3. (i) Let G be the $ax+b$-group, that is,

$$G = \{(a, s) : a, s \in \mathbb{R}, a > 0\}$$

with multiplication $(a, s)(b, t) = (ab, s + at)$. Let H be the subgroup consisting of all $(a, 0)$, $a > 0$ and let $G^+ = \{(a, s) : a > 0, s > 0\}$. Then, for every $t \in \mathbb{R}$, $t > 0$, we have that $H(1, t)\,H = G^+$. Thus, if $\phi \in P_H(G)$, then $\phi(g) = \phi((1, t))$ for all $g \in G^+$ and all $t > 0$. With $t \to 0$, it follows by continuity that ϕ is identically one on G^+.

(ii) Consider the Heisenberg group G. Thus $G = \mathbb{R}^3$ with multiplication

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2),$$

$x_i, y_i, z_i \in \mathbb{R}$, $i=1, 2$. Let $H=\{(x, 0, 0) : x \in \mathbb{R}\}$ and $\phi \in P_H(G)$. It is easily verified that for $g_y=(0, y, 0)$, $y \neq 0$,

$$\{[h, g_y]=hg_yh^{-1}g_y^{-1} : h \in H\} = \{(0, 0, t) : t \in \mathbb{R}\},$$

the centre $Z(G)$ of G. Since

$$\phi(g_y)=\phi(hg_yh^{-1})=\phi([h, g_y]\, g_y),$$

we obtain that $\phi(g_y)=\phi(g_yg)$ for every $g \in Z(G)$ and $y \in \mathbb{R}$, $y \neq 0$. With $y \to 0$, we conclude that $\phi(g)=1$ for all $g \in Z(G)$. This shows that G does not have the separation property.

(iii) With the notation of [1, p. 180–182], let $G=G_{3,4}(\alpha)$, $\alpha \in \mathbb{R}$. Then G can be realized as $\mathbb{R}^3$ with multiplication

$$\begin{aligned}(t_1, x_1, y_1)&(t_2, x_2, y_2)\\ &=(t_1+t_2, x_1+e^{\alpha t_1}(x_2 \cos t_1 - y_2 \sin t_1), y_1\\ &\quad + e^{\alpha t_1}(x_2 \sin t_1 + y_2 \cos t_1)),\end{aligned}$$

$t_j, x_j, y_j \in \mathbb{R}$, $j=1, 2$. Let $H=\{(0, x, 0) : x \in \mathbb{R}\}$, a closed subgroup. Since

$$(0, x_1, 0)(t, 0, 0)(0, x_2, 0)=(t, x_1+x_2e^{\alpha t} \cos t, x_2e^{\alpha t} \sin t),$$

it follows that, for $0<t<\pi$,

$$H(t, 0, 0)\, H=\{(t, x, y) : x, y \in \mathbb{R}\}.$$

Consequently, for each $\phi \in P_H(G)$,

$$\phi((t, 0, 0))=\phi((t, x, y))$$

for all $x, y \in \mathbb{R}$ and $0<t<\pi$. With $t \to 0$, we conclude that $\phi((0, x, y))=1$ for all $x, y \in \mathbb{R}$. Thus the separation property fails for G.

LEMMA 1.4. *Let H be a solvable simply connected Lie group, and suppose that H has the separation property. Then H is abelian.*

Proof. Suppose that the statement of the lemma fails to hold, and let H be a non-abelian, simply connected, solvable Lie group of minimal dimension which has the separation property. We claim that H has a non-abelian, simply connected subgroup G of dimension 2 or 3.

To that end, let V be a non-trivial normal vector subgroup of H of minimal dimension, and let $q: H \to H/V$ denote the quotient homomorphism. It is well-known that V is of dimension 1 or 2. Since H/V has the separation property, by the minimality of H, H/V must be abelian and hence a vector group. Now, if $\dim V = 1$, then choose non-commuting elements x and y in H and let $G = q^{-1}(\mathbb{R}q(x) + \mathbb{R}q(y))$. If $\dim V = 2$, then due to the minimality of V, V cannot be contained in the centre of H. Then choose $x \in V$ and $y \in H$ such that $[x, y] \neq e$, and let $G = q^{-1}(\mathbb{R}q(y))$.

Thus it suffices to show that any non-abelian, simply connected, solvable Lie group G of dimension $\leqslant 3$ does not have the separation property. Now, the non-abelian, solvable, real Lie algebras $\mathfrak{g}$ of dimension $\leqslant 3$ are classified in [1, pp. 180–182]. Retaining the notation of [1], all such $\mathfrak{g}$ except the Heisenberg Lie algebra and $\mathfrak{g}_{3,4}(\alpha)$ contain the Lie algebra of the $ax+b$-group as a subalgebra. Since the separation property is inherited by closed subgroups, it therefore remains to show that none of the Heisenberg group, the $ax+b$-group and $G_{3,4}(\alpha)$, the simply connected Lie group corresponding to $\mathfrak{g}_{3,4}(\alpha)$, does have the separation property. However, this has been verified in Example 1.3. ∎

LEMMA 1.5. *Suppose that G contains a central torus $\mathbb{T}^d$ such that $G/\mathbb{T}^d$ is a vector group. If G has the separation property, then G is a direct product of $\mathbb{T}^d$ and a vector group.*

Proof. By the structure theorem for connected abelian Lie groups, it suffices to show that G is abelian. Let $\mathscr{C}$ be the collection of all closed subgroups C of $\mathbb{T}^d$ such that $\mathbb{T}^d/C = \mathbb{T}$. Since $\bigcap \{C : C \in \mathscr{C}\} = \{1\}$, it is enough to show that each G/C is abelian. Since every G/C has the separation property, we can assume that G contains a central torus $\mathbb{T}$ such that $G/\mathbb{T}$ is a vector group.

Suppose that G is non-abelian. There exists a closed subgroup H of G, containing $\mathbb{T}$, such that $H/\mathbb{T} = \mathbb{R}^2$ and H is non-abelian. Then H is isomorphic to the so-called reduced Heisenberg group, that is, $H = \mathbb{R}^2 \times \mathbb{T}$ with multiplication

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 z_2 e^{2\pi i x_1 y_2}),$$

$x_j, y_j \in \mathbb{R}$, $z_j \in \mathbb{T}$, $j = 1, 2$. For $x, y \in \mathbb{R}$ and $z \in \mathbb{T}$, let

$$a_x = (x, 0, 1), \qquad b_y = (0, y, 1), \qquad \text{and} \qquad c_z = (0, 0, z).$$

Let $L = \{a_x : x \in \mathbb{R}\}$ and $\phi \in P_L(H)$. Then

$$\phi(b_y) = \phi([a_x, b_y]\, b_y) = \phi(c_{e^{2\pi i x y}} b_y),$$

and hence $\phi(b_y) = \phi(c_z b_y)$ for all $z \in \mathbb{T}$ whenever $y \neq 0$. With $y \to 0$, we obtain that $\phi(c_z) = 1$ for all $z \in \mathbb{T}$ (compare the proof for the Heisenberg group). This contradicts the separation property. ∎

The final special case to deal with is that of a semisimple Lie group.

LEMMA 1.6. *Let G be a non-compact connected semisimple Lie group. Then G does not have the separation property.*

Proof. Suppose first that G has finite centre. Let $G = KAN$ be an Iwasawa decomposition of G, M the centralizer of A in K and $H = MAN$ the corresponding minimal parabolic subgroup. Let M^* denote the normalizer of A in K. Then the Weyl group M^*/M is finite. Let W be a coset representative system for M in M^*. Then, by a theorem of Bruhat and Harish-Chandra [28, Theorem 1.2.3.1], $G = \bigcup_{w \in W} HwH$, a finite union.

Now, let $\phi \in P_H(G)$. Then $\phi(w) = \phi(xwy)$ for all $x, y \in H$. Hence ϕ has finite range. Since G is connected, this is impossible unless $\phi \equiv 1$.

Now consider an arbitrary non-compact semisimple Lie group H, and let Z denote the centre of H. Then $G = H/Z$ is a semisimple Lie group with trivial centre. Also, G is non-compact since otherwise G, being a connected group with cocompact centre, has to be the direct product of a compact group and a vector group, contradicting semisimplicity of G. By the first part, the separation property fails for G and hence for H. ∎

We are now ready to prove the implication (i) ⇒ (ii) of Theorem 1.1 for connected Lie groups. In the sequel we shall frequently use, without any further mention, that if H is a locally compact group with the separation property, then closed subgroups and quotient groups of H also have the separation property. First we consider connected solvable Lie groups and argue by induction on the dimension.

Thus let G be a connected solvable Lie group with separation property, and suppose that (i) ⇒ (ii) has already been shown for solvable connected Lie groups of smaller dimension. Let $[G, G]$ denote the closed commutator subgroup of G. Then $[G, G] = W_1 \times C_1$ where W_1 is a vector group and C_1 is a compact connected Lie group. Note that C_1 is normal in G. Also $G/[G, G] = W_2 \times C_2$ where W_2 is a vector group and C_2 is compact. Let H be the closed subgroup of G containing $[\mathrm{G}, \mathrm{G}]$ such that $H/[G, G] = C_2$. Then H/C_1 has a normal subgroup, isomorphic to W_1, with compact connected quotient group C_2. Lemma 1.2 implies that $H/C_1 = W_1 \times K_1$ where K_1 is a compact connected group and K_1 is normal in G. Next, G/K_1 has a normal vector subgroup (namely, W_1) with quotient group W_2. So G/K_1 is simply connected and 2-step solvable. By Lemma 1.4, G/K_1 is a vector group. Since G is a connected Lie group and K_1 is compact, connected and solvable, it follows that K_1 is contained in the centre of G and isomorphic to $\mathbb{T}^d$ for some d. Now, Lemma 1.5 shows that $G = W \times \mathbb{T}^d$ where W is a vector group.

Finally, let G be an arbitrary connected Lie group with separation property, and let R denote the radical of G. Then, by what we have shown above, $R = W \times \mathbb{T}^d$, with W a vector group. Thus R is an SIN-group and every element of R has a compact conjugacy class. By Lemma 1.6, the semisimple group G/R is compact. It follows that every element of R has a compact conjugacy class in G and R has a neighbourhood basis of the identity consisting of G-invariant sets. That is, with the notation of [13], $R \in [SIN]_G \cap [FC]_G^-$, whence by [13, Theorem 1.1] there is a vector subgroup V of G which is normal in G such that $R = V \times \mathbb{T}^d$. So G has a normal vector subgroup V with compact and connected quotient group G/V. Applying Lemma 1.2 again yields that $G = V \times K$ where K is compact group.

This finishes the proof of the theorem.

2. EXAMPLES AND REMARKS

Let G be a locally compact group and H a closed subgroup of G. As mentioned earlier, G has the H-separation property whenever H is normal or open or compact. To capture more subgroups H, recall from [13] that G is said to have small H-invariant neighbourhoods (G belongs to the class $[SIN]_H$) if G has a neighbourhood basis $\mathscr{V}$ of the identity such that $h^{-1}Vh = V$ for all $V \in \mathscr{V}$ and $h \in H$. Then, for instance, $G \in [SIN]_H$ for any locally compact group G and any compact subgroup H of G. More interesting examples can be constructed as in Remark 2.1.

Remark 2.1. Let N be an arbitrary locally compact group and K a compact group of topological automorphisms of N. Then N possesses a neighbourhood basis $\mathscr{V}$ of the identity such that $\tau(V) = V$ for all $V \in \mathscr{V}$ and $\tau \in K$. Let H be any (not necessarily closed) subgroup of K. Endow H with the discrete topology and form the semi-direct product, G, of H and N defined by the action of H on N. Then $G \in [SIN]_H$ since N is open in G.

The condition that $G \in [SIN]_H$ can be further weakened as follows. A closed subgroup H of a locally compact group G is said to be *neutral* in G if for every neighbourhood U of the identity e of G there exists a neighbourhood V of e such that $VH \subseteq HU$ [26]. Actually, if H is neutral in G, then there exists a neighbourhood basis $\mathscr{W}$ of e such that $WH = HW$ for all $W \in \mathscr{W}$. Indeed, if U and V are as above, then $W = U \cap U^{-1} \cap H(V \cap V^{-1})\, H$ satisfies $HW = WH$. This notion of neutral subgroup covers the cases that $G \in [SIN]_H$ and that H is open in its normalizer.

The following proposition extends [11, Proposition 3.10], and the proof is an adaptation of that given in [11] for the SIN-group case. However, for the reader's convenience, we include a sketch of proof.

PROPOSITION 2.2. *Let G be a locally compact group and H a closed neutral subgroup. Then, given any compact subset C of G with $C \cap H = \varnothing$,* *there exists $u \in P_H(G)$ such that $u(x) = 0$ for all $x \in C$. In particular, G has the H-separation property.*

Proof. Notice first that since H is neutral in G, by [25] there exists an invariant measure μ on the left coset space G/H (equivalently, the modular functions of G and of H agree on H). So Weil's formula

$$\int_G f(x)\,dx = \int_{G/H} \int_H f(xh)\,dh\,d\mu(xH),$$

$f \in L^1(G)$, holds. Let T_H denote the map from $L^1(G)$ onto $L^1(G/H, \mu)$ given by $T_H f(xH) = \int_H f(xh)\,dh$.

Now choose a symmetric neighbourhood U of the identity in G such that $UCU \cap H = \varnothing$. Since H is neutral in G, there exists a compact symmetric neighbourhood V of the identity such that $V \subseteq U$ and $HV = VH$. Let $q\colon G \to G/H$ denote the quotient homomorphism and let v be nonnegative function in $L^1(G)$ such that $T_H v = \mu(q(V))^{-1/2}$ on $q(V)$ and $T_H v = 0$ on $G/H \backslash q(V)$. Then $T_H v$ has norm 1 in $L^2(G/H, \mu)$.

Now define u on G by

$$u(x) = \int_G v(y)\, T_H v(x^{-1}yH)\,dy.$$

Then, by the choice of U, V and v, it is easily verified that $u(x) = 0$ for all $x \in C$.

We claim that $u(h) = 1$ for every $h \in H$. To see this, observe that if $y \in V$, then there exist $k \in H$ and $z \in V$ such that $hy = zk$. Hence

$$\int_H v(hyt)\,dt = \int_H v(zs)\,ds = \mu(q(V))^{-1/2}.$$

Since μ is H-invariant, it follows that

$$\begin{aligned} u(h) &= \int_{G/H} \left(T_H v(h^{-1}yH) \int_H v(yt)\,dt \right) d\mu(yH) \\ &= \int_{q(V)} \left(T_H v(yH) \int_H v(hyt)\,dt \right) = 1. \end{aligned}$$

Finally, denoting by π the representation of G induced from the trivial representation of H, the formula for u can be rewritten as

$$u(x) = \int_{G/H} T_H v(x^{-1}yH)\, \overline{T_H v(yH)}\,d\mu(yH) = \langle \pi(x)\, T_H v, T_H v \rangle.$$

Thus u is positive definite, as was to be shown. ∎

LEMMA 2.3. *Let G be a projective limit of groups $G_\alpha = G/K_\alpha$, $\alpha \in A$, and for each $\alpha \in A$, let $q_\alpha : G \to G_\alpha$ denote the quotient homomorphism. Let H be a closed subgroup of G. Then the following conditions are equivalent.*

(i) *G has the H-separation property.*

(ii) *The set $\bigcup_{\alpha \in A} P_{HK_\alpha}(G)$ separates points of $G \backslash H$ from points of H.*

In particular, if G_α has the $q_\alpha(H)$-separation property for every α, then G has the H-separation property.

Proof. Suppose first that (i) holds, and let $x \in G \backslash H$. There exists $\phi \in P_H(G)$ such that $\phi(x) \neq 1$. For each $\alpha \in A$, let μ_α denote normalized Haar measure of K_α and define $\phi_\alpha : G \to \mathbb{C}$ by

$$\phi_\alpha(y) = \int_{K_\alpha} \phi(yt)\, d\mu_\alpha(t).$$

It is straightforward to verify that ϕ_α is a continuous positive definite function. Moreover, for $h \in H$ and $s \in K_\alpha$, since $\phi \in P_H(G)$,

$$\phi_\alpha(hs) = \int_{K_\alpha} \phi(hst)\, d\mu_\alpha(t) = \int_{K_\alpha} \phi(t)\, d\mu_\alpha(t) = \phi_\alpha(e).$$

Now, there exists a neighbourhood V of the identity in G such that

$$|\phi(y) - \phi(z)| < \min(1, \tfrac{1}{2} |1 - \phi(x)|)$$

for all $y, z \in G$ with $z^{-1} y \in V$. Then, for α large enough, $K_\alpha \subseteq V$ and hence

$$\phi_\alpha(e) \geqslant 1 - \int_{K_\alpha} |1 - \phi(t)|\, d\mu_\alpha(t) > 0$$

and also

$$|\phi_\alpha(x) - \phi_\alpha(e)| = \left| \int_{K_\alpha} (\phi(xt) - \phi(t))\, d\mu_\alpha(t) \right| \geqslant |\phi(x) - 1|$$

$$- \int_{K_\alpha} |\phi(xt) - \phi(x)|\, d\mu_\alpha(t) - \int_{K_\alpha} |\phi(t) - 1|\, d\mu_\alpha(t) > 0.$$

Consequently, $\psi_\alpha = \phi_\alpha(e)^{-1} \phi_\alpha \in P_{HK_\alpha}(G)$ and $\psi_\alpha(x) \neq 1$. This proves (ii).

Conversely, suppose that (ii) is satisfied and let $x \in G \backslash H$. There exist $\alpha \in A$ and $\phi_\alpha \in P_{HK_\alpha}(G)$ such that $\phi(x) \neq 1$. Thus (i) holds.

Finally, the last statement follows from

$$P_{HK_\alpha}(G) \supseteq P_{q_\alpha(H)}(G_\alpha) \circ q_\alpha,$$

$\alpha \in A$, and the implication (ii) $\Rightarrow$ (i). ∎

We conclude this section with various remarks concerning the separation property.

Remark 2.4. Let G be a locally compact group and H a closed subgroup of G. A problem of great significance that has been investigated by several authors (see [3, 14, 21, 22] and the references therein) is the question of whether every $\phi \in P(H)$ extends to some $\tilde{\phi} \in P(G)$. For instance, the main result of [14] says that the answer is affirmative whenever $G \in [SIN]_H$. The same result was shown in [3], although not stated in this generality. On the other hand, it was proved in [3] that if G is a connected Lie group such that every closed subgroup of G shares this extension property, then $G \in [SIN]$ (equivalently, G is the direct product of a vector group and a compact group). Thus comparison with Theorem 1.1 shows some unexpected parallel in the type of results holding for the separation and the extension properties, at least for connected Lie groups.

Remark 2.5. Let G be a locally compact group and G_0 the connected component of the identity of G, and suppose that G_0 is the direct product of a vector group and a compact group. Let H be a closed subgroup of G_0. Then G has the H-separation property. To see this, let first $x \in G \backslash G_0$. Then, since G_0 is normal, there exists $\phi \in P_{G_0}(G) \subseteq P_H(G)$ such that $\phi(x) \neq 1$. Secondly, let $x \in G_0 \backslash H$. Then there exists $\psi \in P_H(G_0)$ so that $\psi(x) \neq 1$. By [18, Proposition 1.1], ψ extends to a positive definite functions ϕ on G, and $\phi \in P_H(G)$, $\phi(x) \neq 1$.

Remark 2.6. Let G be a nilpotent locally compact group, and let $\{e\} = Z_0 \subseteq Z_1 \subseteq \ldots$ denote the ascending central series of G. Let H be a closed subgroup of G, and suppose that all subgroups $H_m = \overline{HZ_m}$ $(m \in \mathbb{N}_0)$ have the extension property. Then H has the separation property. For that, let $x \in G \backslash H$ and let m be such that $x \in H_m$, but $x \notin H_{m-1}$. Since H_{m-1} is normal in H_m, there exists $\psi \in P_{H_{m-1}}(H_m) \subseteq P_H(H_m)$ so that $\psi(x) \neq 1$. Now, by hypothesis, ψ admits an extension $\phi \in P(G)$. Thus, in particular, if a nilpotent locally compact group has the extension property, then it has the separation property.

For a locally compact group G, let $B(G)$ denote the Fourier–Stieltjes algebra of G [9]. $B(G)$ consists of all finite linear combinations of continuous positive definite functions and is the dual Banach space of the group C^*-algebra of G (see [9]).

Remark 2.7. For a closed subgroup H of G and $c > 0$, let

$$B_H^c(G) = \{u \in B(G) : u(h) = 1 \text{ for all } h \in H \text{ and } \|u\| \leqslant c\}.$$

Then $B_H^1(G) = P_H(G)$. In fact, given $u \in B_H(G)$, by [9, Lemme 2.14] there exist a unitary representation π of G and ξ and η in the Hilbert space of π such that

$$\|u\| = \|\xi\| \cdot \|\eta\| \qquad \text{and} \qquad u(x) = \langle \pi(x)\, \xi, \eta \rangle$$

for all $x \in G$. Hence, if $u \in B_H^1(G)$, then

$$1 = \langle \xi, \eta \rangle \leqslant \|\xi\| \cdot \|\eta\| = \|u\| \leqslant 1.$$

It follows that $\xi = \eta$, whence $u \in P(G)$.

In view of this it is worth mentioning that $B_H^c(G)$ separates points of $G \backslash H$ from points in H whenever $c > 1$. To see this, notice that given $x \in G \backslash H$, there exists $v \in B(G)$ such that $v(x) \neq 0$, $v(h) = 0$ for all $h \in H$ and $\|v\| \leqslant c - 1$ [9, Lemme 3.2]. Then $u = 1 + v$ belongs to $B_H^c(G)$ and satisfies $u(x) \neq 1$.

3. APPLICATIONS TO FOURIER ALGEBRAS

As outlined in the introduction, in this section we are going to apply the H-separation property to two different problems in the ideal theory of Fourier algebras. We start with a characterization of the H-separation property which is required in both of these applications and also appears to be of interest in its own.

However, we first have to introduce some notation. Let G be a locally compact group, and let ρ_G (or simply ρ) denote the left regular representation of G on $L^2(G)$. The *Fourier algebra* of G, $A(G)$, has been introduced by Eymard [9]. It is the closed ideal of $B(G)$ generated by all compactly supported functions in $B(G)$ and turns out to be just the set of coefficients of ρ [9], that is, $u \in A(G)$ if and only if there are f and g in $L^2(G)$ such that $u(x) = \langle \rho(x) f, g \rangle$ for all $x \in G$. Recall that, when G is abelian, $A(G)$ is isometrically isomorphic (by means of the Fourier transform) to $L^1(\hat{G})$, the L^1-algebra of the dual group $\hat{G}$ of G. The spectrum of $A(G)$ can be identified with G (point evaluations of functions in $A(G)$) [9, Théorème 3.34], and $A(G)$ is regular in the sense that given a compact subset C of G and a closed subset E of G such that $C \cap E = \varnothing$, there exists $u \in A(G) \cap C_c(G)$ such that $u(x) = 1$ for all $x \in C$ and $u(y) = 0$ for all $y \in E$ [9, Lemme 3.2]. For a closed subset E of G, let

$$I(E) = \{u \in A(G) : u(x) = 0 \text{ for all } x \in E\}$$

and

$$J(E) = \{u \in A(G) \cap C_c(G) : u \text{ vanishes on a neighbourhood of } E\}.$$

Then $J(E) \subseteq I \subseteq I(E)$ for every ideal I of $A(G)$ with zero set E.

Let $VN(G)$ denote the closure in the weak operator topology of the linear span of $\{\rho(x): x \in G\}$ in $\mathscr{B}(L^2(G))$, the algebra of bounded linear operators on $L^2(G)$. Then $A(G)$ is the unique predual of the von Neumann algebra $VN(G)$ [9, Théorème 3.10], and for $T \in VN(G)$ and $u \in A(G)$, we write $\langle T, u\rangle$ for the value of T at u. There is a natural action of $A(G)$ (in fact, of $B(G)$) on $VN(G)$ given by

$$\langle v \cdot T, u\rangle = \langle T, vu\rangle, \; T \in VN(G), u \in A(G), v \in B(G).$$

For a closed subgroup H of G, let $VN_H(G)$ denote the von Neumann subalgebra of $VN(G)$ generated by $\{\rho_G(h): h \in H\}$. Then $VN_H(G) = I(H)^\perp$, the annihilator of $I(H)$ in $VN(G)$. Indeed, the inclusion $VN_H(G) \subseteq I(H)^\perp$ is immediate from the definition of $VN_H(G)$ and the fact that $\langle \rho_G(x), u\rangle = u(x)$ for $x \in G$ and $u \in A(G)$. Conversely, if $T \in I(H)^\perp$ then, by [10, Lemma 3.8], $r(u) \to \langle T, u\rangle$, $u \in A(G)$, defines a bounded linear functional on $A(H)$. Thus, for some $S \in VN(H)$,

$$\langle T, u\rangle = \langle S, r(u)\rangle = \langle r^*(S), u\rangle$$

for all $u \in A(G)$, whence $T = r^*(S) \in VN_H(G)$.

It follows from [19, Theorem 2] that if G has the H-separation property, then there exists a continuous projection P from $VN(G)$ onto $VN_H(G)$ such that $P(u \cdot T) = u \cdot P(T)$ for all $T \in VN(G)$ and $u \in A(G)$. For H normal, the existence of such a projection has also been shown by Derighetti [6]. In what follows we need a strengthening of the mere fact that such projections exist.

Let $\mathscr{B}(VN(G))$ denote the space of all bounded linear operators $T: VN(G) \to VN(G)$ equipped with the weak*-operator topology (that is, a net $(\Lambda_\alpha)_\alpha$ in $\mathscr{B}(VN(G))$ converges to Λ if and only if $\langle \Lambda_\alpha(T), u\rangle \to \langle \Lambda(T), u\rangle$ for all $T \in VN(G)$ and $u \in A(G)$). Then $\mathscr{B}(VN(G))_1$, the unit ball of $\mathscr{B}(VN(G))$, is compact.

For $u \in B(G)$, define $\Lambda_u \in \mathscr{B}(VN(G))$ by $\Lambda_u(T) = u \cdot T$. Let

$$\mathscr{K}_H = \overline{\{\Lambda_u : u \in P_H(G)\}},$$

the closure in the weak*-operator topology. Then $\mathscr{K}_H$ is a compact convex subset of $\mathscr{B}(VN(G))_1$, and $\Lambda(u \cdot T) = u \cdot \Lambda(T)$ for each $\Lambda \in \mathscr{K}_H$, $u \in P_H(G)$ and $T \in VN(G)$.

PROPOSITION 3.1. *Let G be a locally compact group and H a closed subgroup of G. Then the following three conditions are equivalent.*

(i) *G has the H-separation property.*

(ii) *There exists a projection P from $VN(G)$ onto $VN_H(G)$ such that $P \in \mathscr{K}_H$.*

(iii) $VN_H(G) = \{T \in VN(G) : u \cdot T = T \text{ for all } u \in P_H(G)\}$.

Every projection P as in (ii) *has norm* 1.

Proof. (i) $\Rightarrow$ (ii). For each $u \in P_H(G)$, let $\phi_u\colon \mathscr{K}_H \to \mathscr{B}(VN(G))$ be defined by

$$\phi_u(\Lambda)(T) = u \cdot \Lambda(T), \quad T \in VN(G).$$

If $\Lambda \in \mathscr{K}_H$, then there exists a net $(u_\alpha)_\alpha$ in $P_H(G)$ such that $\Lambda_{u_\alpha} \to \Lambda$ in the w^*-operator topology. Now $\phi_u(\Lambda_{u_\alpha}) = \Lambda_{uu_\alpha}$ converges to $\phi_u(\Lambda)$ in the w^*-operator topology, whence $\phi_u(\Lambda) \in \mathscr{K}_H$. Consequently, $\{\phi_u : u \in P_H(G)\}$ is a commuting family of continuous affine maps from $\mathscr{K}_H$ into $\mathscr{K}_H$. Thus, by the Markov-Kakutani fixed point theorem [8, p. 456, Theorem 6], there exists $P \in \mathscr{K}_H$ such that $\phi_u(P) = P$ for all $u \in P_H(G)$. Now, by [19, Lemma 6] (and its proof), $P(T) \in VN_H(G)$ for all $T \in VN(G)$ and $P(T) = T$ for all $T \in VN_H(G)$.

(ii) $\Rightarrow$ (iii). Let $(u_\alpha)_\alpha$ be a net in $P_H(G)$ such that

$$\langle u_\alpha \cdot T, u\rangle \to \langle P(T), u\rangle$$

for all $T \in VN(G)$ and $u \in A(G)$. Suppose that $T \in VN(G)$ is such that $u \cdot T = T$ for all $u \in P_H(G)$. It follows that $T = P(T) \in VN_H(G)$. On the other hand, for all $h \in H$, $u \in P_H(G)$ and $v \in A(G)$,

$$\langle u \cdot \rho_G(h), v\rangle = \langle \rho_G(h), uv\rangle = u(h)\, v(h) = v(h) = \langle \rho_G(h), v\rangle,$$

and hence $u \cdot \rho_G(h) = \rho_G(h)$ for all $u \in P_H(G)$ and $h \in H$. This implies that $u \cdot T = T$ for every $T \in VN_H(G)$ and $u \in P_H(G)$.

(iii) $\Rightarrow$ (i). Let $x \in G$ be such that $u(x) = 1$ for all $u \in P_H(G)$. Then, for all $v \in A(G)$ and $u \in P_H(G)$,

$$\langle u \cdot \rho_G(x), v\rangle = (uv)(x) = \langle \rho_G(x), v\rangle.$$

Thus $u \cdot \rho_G(x) = \rho_G(x)$, whence $\rho_G(x) \in VN_H(G)$ by condition (iii). By definition of $VN_H(G)$, this implies that $x \in H$. So G has the H-separation property.

Finally, let P be as in (ii). Since $|\langle \Lambda_u(T), v \rangle| = |\langle T, uv \rangle| \leqslant \|T\| \, \|u\| \, \|v\|$ for all $T \in VN(G)$ and $u, v \in A(G)$, it follows that each Λ_u, $u \in P_H(G)$, has norm one, and hence so does P. ∎

In particular, the implication (iii) $\Rightarrow$ (i) of the preceding proposition shows that [19, Lemma 6] does not hold for closed subgroups in general.

Let A be a commutative Banach algebra. Recall that an approximate identity for A with norm bound $c > 0$ is a net $(u_\alpha)_\alpha$ in A such that $\|u_\alpha\| \leqslant c$ for all α and $\|u_\alpha a - a\| \to 0$ for every $a \in A$.

Several authors have investigated the problem of which closed ideals of $A(G)$ have bounded approximate identities [4, 10, 11, 20]. The starting point has been Leptin's theorem [20] that $A(G)$ itself has a bounded approximate identity (of norm 1) precisely when G is amenable. However, proper closed ideals of $A(G)$ very seldom have approximate identities with norm bound 1. Indeed, for any closed subset E of G, the ideal $I(E)$ admits an approximate identity with norm bound 1 if and only if $G \backslash E$ is a coset of some open amenable subgroup of G [10, Proposition 3.12]. It turns out that if H is a non-open closed subgroup of G, then 2 is the best possible norm bound for an approximate identity of $I(H)$ (Proposition 3.3 below). This generalizes [4, Theorem 10]. On the other hand, it is not unlikely that $I(H)$ has a bounded approximate identity for any closed subgroup H of an amenable group G. Such a conjecture is supported by results from [10, 11], where it was shown to be true whenever H is compact or open or normal in G, or if G is an SIN-group. In each of these cases the estimate of [10, Proposition 3.2] provides the norm bound 3. Our first goal is to improve these results by showing that if G is amenable and H is a closed subgroup of G such that G has the H-separation property, then $I(H)$ has a bounded approximate identity with norm bound 2.

Lemma 3.2. *Let H be a non-open closed subgroup of G. Let $P\colon VN(G) \to I(H)^\perp$ be a projection, and suppose there exists a bounded net $(u_\alpha)_\alpha$ in $B(G)$ such that*

$$\langle u_\alpha \cdot T, u \rangle \to \langle P(T), u \rangle$$

for all $T \in VN(G)$ and all $u \in A(G)$. Then $P(T) = 0$ for all $T \in C^*_\rho(G)$.

Proof. To establish the lemma, it suffices to prove that $P(\rho_G(g)) = 0$ for every $g \in C_c(G)$. For a measurable subset M of G, let $|M|$ denote the Haar measure of M. Since H is not open in G, $|H| = 0$, and hence given $\varepsilon > 0$, there is an open subset V containing $H \cap \operatorname{supp} g$ such that $|V| \leqslant \varepsilon$. Let $f = g \cdot 1_{G \setminus V}$. Then

$$\|P(\rho_G(g)) - P(\rho_G(f))\| \leqslant \|\rho_G(g-f)\| \leqslant \|g-f\|_1$$
$$\leqslant |V| \; \|g\|_\infty \leqslant \varepsilon \; \|g\|_\infty .$$

We now show that $P(\rho_G(f)) = 0$. For that, let $K = \operatorname{supp} g \cap (G \setminus V)$, a compact subset of $G \setminus H$. Then

$$\int_K (fu)(x)\, u_\alpha(x)\, dx = \int_G (fu)(x)\, u_\alpha(x)\, dx$$
$$= \langle u_\alpha \cdot \rho_G(f), u \rangle \to \langle P(\rho_G(f)), u \rangle$$

for all $u \in A(G)$. Since $A(G)$ is regular, there exists $v \in A(G)$ such that $v(x) = 0$ for all $x \in H$ and $v(x) = 1$ for all $x \in K$. Then, since $vu \in I(H)$ and $P(\rho_G(f)) \in I(H)^\perp$,

$$\int_K (fu)(x)\, u_\alpha(x)\, dx = \int_K f(x)\, vu(x)\, u_\alpha(x)\, dx \to \langle P(\rho_G(f)), vu \rangle = 0.$$

Hence $\langle P(\rho_G(f)), u \rangle = 0$ for all $u \in A(G)$, as required. ∎

PROPOSITION 3.3. *Let G be a locally compact group and H a non-open closed subgroup of G. Then 2 is the best possible norm bound for an approximate identity of $I(H)$.*

Proof. Suppose there exists an approximate identity $(v_\alpha)_\alpha$ in $I(H)$ such that, for some constant $c < 2$, $\|v_\alpha\| \leqslant c$ for all α. Then, after passing to a subnet if necessary, we can assume that $(v_\alpha)_\alpha \subseteq VN(G)^*$ converges in the w^*-topology of $VN(G)^*$. As shown in the proof of Proposition 6.4 of [10], this gives rise to a projection P from $VN(G)$ onto $I(H)^\perp$ such that

$$\langle (1 - v_\alpha) \cdot T, u \rangle \to \langle P(T), u \rangle$$

for all $T \in VN(G)$ and $u \in A(G)$. Lemma 3.2 now yields $P(C_\rho^*(G)) = \{0\}$.

Let I and $I_{L^2(G)}$ denote the identity of $\mathcal{B}(VN(G))$ and $\mathcal{B}(L^2(G))$, respectively. By [4, Proposition 9] there exists $T \in C_\rho^*(G)$ such that $\|T\| = 1$ and $\|I_{L^2(G)} - 2T\| = 1$. Since $P(T) = 0$, it follows that (compare the proof of [4, Theorem 10])

$$(I - P)(I_{L^2(G)} - 2T) = -2T,$$

whence $\|I-P\| \geqslant 2$. Thus, since $c<2$, there exist $S \in VN(G)$ and $u \in A(G)$ such that $\|S\| \leqslant 1$, $\|u\| \leqslant 1$ and

$$|\langle S-P(S), u\rangle| \geqslant 1+\frac{c}{2}.$$

Then, by definition of P, $|\langle v_\alpha \cdot S, u\rangle| > c$ for sufficiently large α. This contradicts $|\langle v_\alpha \cdot T, u\rangle| \leqslant \|v_\alpha\| \cdot \|T\| \cdot \|u\| \leqslant c$. ∎

THEOREM 3.4. *Let G be an amenable locally compact group and H a closed subgroup of G such that G has the H-separation property. Then the ideal $I(H)$ of $A(G)$ has an approximate identity with norm bound 2. Moreover, if G/H is infinite, then 2 is the best possible norm bound for an approximate identity of $I(H)$.*

Proof. Since G is amenable, $A(G)$ has an approximate identity with norm bound equal to 1 [20]. Let P: $VN(G) \to VN_H(G) = I(H)^\perp$ be a projection as in Proposition 3.1. Since P has norm 1, by Proposition 2 of [4] for any $v \in I(H)$ and $\epsilon > 0$, there exists $u \in I(H)$ auch that $\|u\| \leqslant 2$ and $\|uv - v\| \leqslant \epsilon$. It was independently shown by Altman and Wichmann (see [24, Theorem 5.1.2(c)]) that this implies the existence of an approximate identity with norm bound 2.

The second statement of the theorem follows from Proposition 3.3 whenever H is not open in G. Finally, if H is open in G and of infinite index, then 2 is the best possible norm bound by [4, Theorem 11]. ∎

Let X be an $A(G)$-invariant linear space of $VN(G)$. A closed subset E of G is called an *X-Ditkin set* for $A(G)$ if for every $T \in X$ and $u \in I(E)$ there exists a net $(v_\alpha)_\alpha$ in $J(E)$ such that

$$\langle v_\alpha \cdot T, u\rangle \to \langle T, u\rangle.$$

Equivalently, given $T \in X$ and $u \in I(E)$, there exists $v \in J(E)$ such that $\langle T, u\rangle = \langle T, vu\rangle$. This notion of X-Ditkin set has been introduced in [17]. When specializing to $X = VN(G)$ and $X = UC_c(\hat{G})$, the set of operators in $VN(G)$ with compact support, one obtains the classical notions of Ditkin set and local Ditkin set, respectively (see [5, Proposition 9; 17, Lemma 2.6]).

Now let H be a closed subgroup of G. Our second application of the H-separation property concerns an injection theorem for X-Ditkin sets. Let

$$r: A(G) \to A(H),\ u \to r(u)$$

be the restriction map. r is norm decreasing and surjective, and hence the adjoint map

$$r^*: VN(H) \to VN(G), \langle r^*(S), u \rangle = \langle S, r(u) \rangle,$$

$u \in A(G)$, $S \in VN(H)$, is injective. In fact, r^* is a weak*-weak*-continuous isomorphism from $VN(H)$ onto $VN_H(G)$ and it maps $UC_c(\hat{H})$ onto $UC_c(\hat{G}) \cap VN_H(G)$ [17, Lemmas 3.1 and 3.2]. For any $A(G)$-invariant subspace X of $VN(G)$, let

$$X_H = r^{*-1}(X).$$

Then X_H is an $A(H)$-invariant subspace of $VN(H)$. Indeed, if $S \in X_H$ and $v \in A(H)$ and $u \in A(G)$ such that $r(U) = v$, then

$$r^*(v \cdot S) = u \cdot r^*(S) \in X,$$

hence $v \cdot S = r^{*-1}(u \cdot r^*(S))$.

THEOREM 3.5 (Injection theorem for X-Ditkin sets). *Let G be a locally compact group and X an $A(G)$-invariant linear subspace of $VN(G)$. Let H be a closed subgroup of G and E a closed subset of H.*

(i) *If E is X-Ditkin for $A(G)$, then E is X_H-Ditkin for $A(H)$.*

(ii) *Suppose that there is a projection P from $VN(G)$ onto $VN_H(G)$ such that $P \in \mathscr{K}_H$ and $P(X) \subseteq X$. In addition, suppose that for every $T \in X$ and $u \in I(H)$,*

$$\langle T, u \rangle \in \{\langle T, vu \rangle : v \in A(G)\}.$$

Then, if E is X_H-Ditkin for $A(H)$, it is also X-Ditkin for $A(G)$.

Proof. Let $I_H(E) = \{w \in A(H) : w(x) = 0 \text{ for all } x \in E\}$ and

$$J_H(E) = \{w \in A(H) \cap C_c(H) : w \text{ vanishes on a neighbourhood of } E \text{ in } H\}.$$

(i) Let $S \in X_H = r^{*-1}(X)$ and $w \in I_H(E)$. Since r is surjective, there exist $u \in I(E)$ such that $r(u) = w$. Since $r^*(S) \in X$ and E is X-Ditkin, there exists $v \in J(E)$ such that $\langle r^*(S), u \rangle = \langle r^*(S), vu \rangle$. Thus

$$\langle S, w \rangle = \langle r^*(S), u \rangle = \langle r^*(S), vu \rangle = \langle S, wr(v) \rangle.$$

As $r(v) \in J_H(E)$, this shows that E is X_H-Ditkin.

(ii) Let $T \in X$ and $u \in I(E)$. Then $P(T) \in VN_H(G) = I(H)^{\perp}$, and since r^* is an isomorphism of $VN(H)$ onto $VN_H(G)$ [17, Lemma 3.1], there exists $S \in VN(H)$ such that $P(T) = r^*(S)$. By hypothesis, $P(X) \subseteq X$, and hence $S = r^{*-1}(P(T)) \in X_H$. Thus, since E is X_H-Ditkin, there exists $w_1 \in J_H(E)$ such that

$$\langle P(T), u\rangle = \langle r^*(S), u\rangle = \langle S, r(u)\rangle = \langle S, r(u)\, w_1\rangle.$$

Choose $w \in J(E)$ such that $r(w) = w_1$. Then

$$\langle P(T), u\rangle = \langle S, r(uw)\rangle = \langle P(T), uw\rangle. \tag{1}$$

Since $P \in \mathscr{K}_H$, given $\varepsilon > 0$, there exists $p \in P_H(G)$ so that

$$|\langle p \cdot T, u\rangle - \langle P(T), u\rangle| \leqslant \varepsilon \tag{2}$$

and

$$|\langle p \cdot T, uw\rangle - \langle P(T), uw\rangle| \leqslant \varepsilon. \tag{3}$$

Let $s = (1-p)(u - uw)$. Then $s \in I(H)$ and hence, by hypothesis, there exists $t \in A(G)$ such that $\langle T, s\rangle = \langle T, st\rangle$. Now let

$$v = w + (1-p)(1-w)\, t \in J(E) + I(H).$$

Since H is a spectral set [27, Theorem 3], $I(H) = \overline{J(H)} \subseteq \overline{J(E)}$. Hence $v \in \overline{J(E)}$ and, by the definition of s and v,

$$\begin{aligned}\langle T, uw + s\rangle &= \langle T, uw + st\rangle \\ &= \langle T, uw + u(1-p)(1-w)\, t\rangle = \langle T, uv\rangle.\end{aligned} \tag{4}$$

Finally, using (1), (2), (3), and (4), we get

$$\begin{aligned}|\langle T, u\rangle - \langle T, uv\rangle| &= |\langle T, u\rangle - \langle T, uv + s\rangle| \\ &= |\langle T, u\rangle - (\langle T, uw\rangle + \langle T, (1-p)(u - uw)\rangle)| \\ &= |\langle T, u\rangle - \langle p \cdot T, uw\rangle - \langle T, (1-p)\, u\rangle| \\ &= |\langle p \cdot T, u\rangle - \langle p \cdot T, uw\rangle| \\ &\leqslant |\langle p \cdot T, u\rangle - \langle P(T), u\rangle| \\ &\quad + |\langle p(T), uw\rangle - \langle p \cdot T, uw\rangle| \leqslant 2\varepsilon.\end{aligned}$$

Since $v \in \overline{J(E)}$ and $\varepsilon > 0$ was arbitrary, we conclude that $\langle T, u\rangle \in \langle T, uJ(E)\rangle$. This proves that E is X-Ditkin. ∎

In addition to $UC_c(\hat{G})$, we now introduce some more subspaces of $VN(G)$ to which Theorem 3.5 applies. In [12], $UC(\hat{G})$ was defined to be the closed linear span of $\{u \cdot T : u \in A(G), T \in VN(G)\}$. Alternatively, $UC(\hat{G})$ can be defined as the norm closure of $UC_c(\hat{G})$ in $VN(G)$. When G

is abelian, $UC(\hat{G})$ is the C^*-algebra of bounded uniformly continuous functions on the dual group $\hat{G}$ of G (whence the notation in the general case). The collection of operators T in $VN(G)$ for which the set $\{u \cdot T: u \in P(G) \cap A(G), u(e) = 1\}$ is relatively norm compact (weakly compact) is denoted $AP(\hat{G})$ ($WAP(\hat{G})$). Then both $AP(\hat{G})$ and $WAP(\hat{G})$ are closed $A(G)$-invariant subspaces of $VN(G)$. When G is abelian, then $AP(\hat{G})$ and $WAP(\hat{G})$ can be identified with the space of continuous almost periodic and continuous weakly almost periodic functions on $\hat{G}$, respectively. For a discussion of these subspaces of $VN(G)$ see [2, 12, and 18]. If H is a closed subgroup of G, then $r^*(UC(\hat{H})) = UC(\hat{G}) \cap VN_H(G)$, and similarly for $AP(\hat{H})$ and $WAP(\hat{H})$ [17, Lemma 3.2].

Note that if X is either $UC_c(\hat{G})$ or $UC(\hat{G})$, then $P(X) \subseteq X$ for any projection P in $\mathscr{K}_H$. Indeed, if $T \in UC_c(\hat{G})$, then $T = v \cdot T$ for some $v \in A(G) \cap C_c(G)$ and hence

$$P(T) = P(v \cdot T) = v \cdot P(T) \in UC_c(\hat{G})$$

because v has compact support. Since $UC_c(\hat{G})$ is dense in $UC(\hat{G})$, it follows also that $P(UC(\hat{G})) \subseteq UC(\hat{G})$. Also, if $X = AP(\hat{G})$ or $WAP(\hat{G})$, then $P(X) \subseteq X$. Indeed, this is straightforward from the fact that $P(u \cdot T) = u \cdot P(T)$ for all $T \in VN(G)$ and $u \in A(G)$.

As a first consequence we obtain the following generalization of Derighetti's injection theorem [5, Théorème 12] for local Ditkin sets of $A(G)$.

COROLLARY 3.6. *Let G be a locally compact group and H a closed subgroup of G, and suppose that G has the H-separation property. Let E be a closed subset of H. Then E is a local Ditkin set for $A(G)$ if and only if E is a local Ditkin set for $A(H)$.*

Proof. Recall that if $T \in UC_c(\hat{G})$, then $T = v \cdot T$ for some $v \in A(G)$, and hence $\langle T, u \rangle = \langle T, vu \rangle$ for all $u \in A(G)$. The statement now follows from Proposition 3.1, Theorem 3.5, and the above remarks. ∎

COROLLARY 3.7. *Let G be a locally compact group and H a closed subgroup of G. Suppose that G has the H-separation property and that $u \in \overline{uA(G)}$ for all $u \in I(H)$. Let E be a closed subset of H. Then E is Ditkin ($UC(\hat{G})$-Ditkin; $AP(\hat{G})$-Ditkin; $WAP(\hat{G})$-Ditkin) for $A(G)$ if and only if E is Ditkin ($UC(\hat{H})$-Ditkin; $AP(\hat{H})$-Ditkin; $WAP(\hat{H})$-Ditkin) for $A(H)$.*

Proof. We have to show that, given $T \in X$ and $u \in I(H)$, there exists $v \in A(G)$ such that $\langle T, u \rangle = \langle T, vu \rangle$. If $\langle T, u \rangle = 0$, let $v = 0$, and if $\langle T, u \rangle \neq 0$, then notice that since the function $v \to \langle T, vu \rangle$ on $A(G)$ is linear and $u \in \overline{uA(G)}$, the range of this function equals $\mathbb{C}$. The statement now follows from Proposition 3.1, Theorem 3.5, and the above remarks. ∎

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